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Dual Representation as Stochastic Differential Games of Backward Stochastic Differential Equations and Dynamic Evaluations*

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Abstract

In this Note, assuming that the generator is uniform Lipschitz in the unknown variables, we relate the solution of a one dimensional backward stochastic differential equation with the value process of a stochastic differential game. Under a domination condition, an $\mathcal{F}$-consistent evaluations is also related to a stochastic differential game. This relation comes out of a min-max representation for uniform Lipschitz functions as affine functions. The extension to reflected backward stochastic differential equations is also included.

1 Introduction

Let $(\emptyset, \mathcal{F}, P)$ be a probability space, and $\{B_s; s \geqslant 0\}$ a d-dimensional Brownian motion defined on $(\emptyset, \mathcal{F}, P)$. Let $\mathcal{F}_t$ be the σ-algebra generated by $\{B_s; 0 \leqslant s \leqslant t\}$ and the totality of P-null sets in $\mathcal{F}$, $L^2(\mathcal{F}_t)$ the set of all $\mathcal{F}_t$-measurable random variables X such that $E|X|^2 < \infty$, and $\mathcal{L}^2_{\mathcal{F}}(0,T)$ the set of $\mathcal{F}_t$-adapted processes φ such that $E\int_0^T |\varphi|^2\,dt < \infty$. Denote by $\mathcal{T}_t$ the set of all $\mathcal{F}_s$-stopping times taking values in $[t,T]$.

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Consider the following one dimensional backward stochastic differential equation (BSDE):

$$\begin{cases} dy_s &= -f(s, y_s, z_s)\, ds + \langle z_s, dB_s\rangle, \quad 0 \leqslant s \leqslant T; \\ y_T &= \xi \in L^2(\mathcal{F}_T). \end{cases} \tag{1}$$

It is known that when the generator is convex or concave with respect to the unknown variables, BSDE (1) is related with a stochastic control problem.

More precisely, assume that f is concave in the last two variables. Consider the Fenchel-Legender transformation:

$$F(\text{ø}, t, \beta_1, \beta_2) := \sup_{(y,z)}[f(\text{ø}, t, y, z) - \beta_1 y - \langle \beta_2, z\rangle] \tag{2}$$

for any $(\text{ø}, t, \beta_1, \beta_2) \in \text{Ø} \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$. Define

$$D_t^F(\text{ø}) = \{(\beta_1, \beta_2) \in \mathbb{R} \times \mathbb{R}^d : F(\text{ø}, t, \beta_1, \beta_2) < \infty\}. \tag{3}$$

Then the set D_t^F is a.s. bounded. It follows from well-known results (see, e.g., [4]) that

$$f(\text{ø}, t, y, z) = \inf_{(\beta_1,\beta_2)\in D_t^F(\text{ø})}[F(\text{ø}, t, \beta_1, \beta_2) + \beta_1 y + \langle \beta_2, z\rangle], \tag{4}$$

and the infimum is achieved. Let us now denote by $\mathcal{A}$ the set of bounded progressively measurable $\mathbb{R}\times\mathbb{R}^d$ valued processes $\{\beta_1(t), \beta_2(t)) : 0 \leqslant t \leqslant T\}$ such that

$$E\int_0^T F(t, \beta_1(t), \beta_2(t))^2\, dt < \infty. \tag{5}$$

To each $(\beta_1, \beta_2) \in \mathcal{A}$, we associate the unique adapted solution $(Y^{\beta_1,\beta_2}, Z^{\beta_1,\beta_2})$ of BSDE (1) with the coefficient f being replaced with the affine one $f^{\beta_1,\beta_2}(t, y, z) := F(t, \beta_1(t), \beta_2(t)) + \beta_1(t)y + \langle \beta_2(t), z\rangle$. In [4, pages 35–37], the solution y of BSDE (1) is interpreted as the value process of a control problem. That is,

$$y_t = \operatorname*{essinf}_{(\beta_1,\beta_2)\in\mathcal{A}} E[\Phi(t, \beta_1, \beta_2)|\mathcal{F}_t] \tag{6}$$

where

$$\Phi(t, \beta_1, \beta_2) := \Delta_{t,T}^{\beta_1,\beta_2}\xi + \int_t^T \Delta_{t,s}^{\beta_1,\beta_2} F(s, \beta_1(s), \beta_2(s))\, ds \tag{7}$$

and for each $t \in [0,T]$, $\{\Delta_{t,s}^{\beta_1,\beta_2} : t \leqslant s \leqslant T\}$ is the unique solution of the following stochastic differential equation (SDE):

$$d\Delta_{t,s} = \Delta_{t,s}[\beta_1(s)\,ds + \langle \beta_2(s), dB_s\rangle], \quad s \in [t,T]; \quad \Delta_{t,t} = 1. \tag{8}$$

The purpose of this Note is to obtain a similar dual representation for the solution y of BSDE (1) under the Lipschitz assumption on the generator, instead of the convexity assumption on the generator f.

Assume throughout the rest of the Note that there is a constant $C > 0$ such that

$$\left\{\begin{array}{ll} \text{(B1)} & f(\cdot, y, z) \in \mathcal{L}^2_{\mathcal{F}}(0,T) \text{ for any pair } (y,z) \in \mathbb{R} \times \mathbb{R}^d; \\ \text{(B2)} & |f(t,y_1,z_1) - f(t,y_2,z_2)| \leqslant C(|y_1 - y_2| + |z_1 - z_2|) \\ & \text{for any } t \in [0,T] \text{ and } (y_1,z_1),(y_2,z_2) \in \mathbb{R} \times \mathbb{R}^d. \end{array}\right. \tag{9}$$

Then, for any $X \in L^2(\mathcal{F}_t)$, there is unique adapted solution $\{(Y_s, Z_s); 0 \leqslant s \leqslant t\}$ of BSDE (1) with the terminal condition: $Y_t = X$. Define $\mathcal{E}^f_{s,t}[X] := Y_s$ for any $s \in [0,t]$.

The rest of this Note is organized as follows. In section 2, we give a Min-Max representation of a Lipschitz function in terms of affine functions, which is the basis of the Note. In Section 3, we present the dual formula for the solution of one dimensional BSDE (1). In Section 4, the formula obtained in Section 3 is applied to the dynamical evaluation and a dual formula is therefore derived for an $\mathcal{F}_t$-consistent evaluation. Finally in Section 5, a dual formula is also obtained for one dimensional reflected backward stochastic differential equations (RBSDEs) (20).

2 Min-max representation of a Lipschitz function as affine functions

The following representation is due to Evans and Souganidis [2, pages 786–787].

Lemma 2.1. *Let $f : [0,T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. That is, there is a constant $C > 0$ such that*

$$|f(t,x_1) - f(t,x_2)| \leqslant C|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^n. \tag{10}$$

Then for each $t \in [0,T]$ and $x \in \mathbb{R}^n$,

$$f(t,x) = \max_{z \in \mathbb{R}^n} \min_{y \in \overline{O}_n(0,1)} \{C\langle y,x\rangle + F(t,y,z)\} \tag{11}$$

where $F(t,y,z) := f(t,z) - C\langle y, z\rangle$ *for* $y, z \in \mathbb{R}^n$ *and* $\overline{O}_n(0,1)$ *is the closed unit ball in* $\mathbb{R}^n$*.*

Proof. In view of the assumption (10), we have for any $x \in \mathbb{R}^n$

$$\begin{aligned} f(t,x) = & \max_{z\in\mathbb{R}^n}\{f(t,z) - C|x-z|\} \\ = & \max_{z\in\mathbb{R}^n} \min_{y\in\overline{O}_n(0,1)} \{f(t,z) + C\langle y, x-z\rangle\}. \end{aligned}$$

□

Remark 1. *See Fleming [5, pages 996–1000] or Evans [1] for other, more complicated ways of writing a nonlinear function as the max-min (or min-max) of affine mappings.*

3 Backward stochastic differential equations and related stochastic differential games

Denote $(\mathcal{L}^2_{\mathcal{F}}(0,T))^{d+1}$ by $\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^{d+1})$, and by V_{d+1} the subset of $\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^{d+1})$ whose element takes values in the closed unit ball $\overline{O}_{d+1}(0,1)$.

Define the function $F : \emptyset \times [0,T] \times \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R}$ as follows:

$$F(\emptyset, s, \beta_1, \beta_2, \alpha_1, \alpha_2) = f(\emptyset, s, \alpha_1, \alpha_2) - C\beta_1\alpha_1 - C\langle\beta_2, \alpha_2\rangle \qquad (12)$$

for any $(\emptyset, s, \beta_1, \beta_2, \alpha_1, \alpha_2) \in \emptyset \times [0,T] \times \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$. Then, in view of Lemma 2.1, we have for any $(\emptyset, s, \beta_1, \beta_2, \alpha_1, \alpha_2) \in \emptyset \times [0,T] \times \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$,

$$f(\emptyset, t, y, z) = \max_{\alpha\in\mathbb{R}^{d+1}} \min_{\beta\in\overline{O}_{d+1}(0,1)} [F(\emptyset, t, \beta, \alpha) + C\beta_1 y + C\langle\beta_2, z\rangle]. \quad (13)$$

Given $\alpha \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^{d+1})$ and $\beta \in V_{d+1}$, consider the related BSDE:

$$\left\{ \begin{array}{rcl} dY_s &=& -[C\beta_1(s)Y_s + C\langle\beta_2(s), Z_s\rangle \\ && +F(s, \beta_1(s), \beta_2(s), \alpha_1(s), \alpha_2(s))]\, ds + \langle Z_s, dB_s\rangle; \\ Y_T &=& \xi \in L^2(\mathcal{F}_T). \end{array} \right. \qquad (14)$$

The solution is denoted by $(Y^{\alpha,\beta}, Z^{\alpha,\beta})$ when it is necessary to emphasize the dependence on (α, β) with $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$.

Introduce the following stochastic differential equation (SDE):

$$d\Gamma_{t,s} = \Gamma_{t,s}[C\beta_1(s)\,ds + C\langle \beta_2(s), dB_s\rangle], \quad s \in [t,T]; \quad \Gamma_{t,t} = 1. \quad (15)$$

Its solution is denoted by $\Gamma^{\beta}_{t,s}, t \leqslant s \leqslant T$ to indicate the dependence on $\beta = (\beta_1, \beta_2)$.

We have

$$Y_t^{\alpha,\beta} = E\left[\int_t^T \Gamma^{\beta}_{t,s} F(s, \beta_1(s), \beta_2(s), \alpha_1(s), \alpha_2(s))\,ds + \Gamma^{\beta}_{t,T}\xi \;\middle|\; \mathcal{F}_t\right] \quad (16)$$

for any $t \in [0,T]$.

Theorem 3.1. *Assume that the function f satisfies (9). Let (y,z) be the adapted solution of BSDE (1) and $\{\Gamma^{\beta}_{t,s}; t \leqslant s \leqslant T\}$ the solution of SDE (15). Then we have for any $t \in [0,T]$,*

$$\begin{aligned} y_t \;=\; & \operatorname*{esssup}_{\alpha \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^{d+1})} \operatorname*{essinf}_{\beta \in V_{d+1}} E\Bigg[\int_t^T \Gamma^{\beta}_{t,s} F(s, \beta_1(s), \beta_2(s), \alpha_1(s), \alpha_2(s))\,ds \\ & \qquad + \Gamma^{\beta}_{t,T}\xi \;\Bigg|\; \mathcal{F}_t\Bigg]. \end{aligned} \quad (17)$$

4 An $\mathcal{F}_t$-consistent evaluations and its dual representation as a stochastic differential game

Définition 4.1. *A system of operators $\mathcal{E}_{s,t} : L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s), 0 \leqslant s \leqslant t \leqslant T$ is called an $\mathcal{F}_t$-consistent evaluation defined on $[0,T]$ if it satisfies the following four properties: for any $0 \leqslant s \leqslant t \leqslant T$ and any $X_1, X_2 \in L^2(\mathcal{F}_t)$,*

(A1) $\mathcal{E}_{s,t}[X_1] \geqslant \mathcal{E}_{s,t}[X_2]$, a.s. , *if* $X_1 \geqslant X_2$, a.s. ;

(A2) $\mathcal{E}_{t,t}[X_1] = X_1$, a.s. ;

(A3) $\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[X_1]] = \mathcal{E}_{r,t}[X_1]$, a.s. ;

(A4) $\chi_A \mathcal{E}_{s,t}[X_1] = \chi_A \mathcal{E}_{s,t}[\chi_A X_1]$, a.s. *for any* $A \in \mathcal{F}_s$.

In view of Peng [7, Corollary 4.2, page 588], the following is an immediate consequence of Theorem **??**.

Theorem 4.1. *Let $\{\mathcal{E}_{s,t}\}_{0\leqslant s\leqslant t\leqslant T}$ denote an $\mathcal{F}_t$-consistent evaluation defined on $[0,T]$. Assume that there is a function $g_\mu(t,y,z) := \mu(|y|+|z|), (t,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d$ for some $\mu > 0$ such that the $\mathcal{F}_t$-consistent evaluation $\{\mathcal{E}_{s,t}\}_{0\leqslant s\leqslant t\leqslant T}$ is dominated by $\mathcal{E}^{g_\mu}_{s,t}$ in the following sense: for any $s,t\in[0,T]$ such that $s\leqslant t$ and for any $X_1, X_2 \in \mathcal{L}^2(\mathcal{F}_t)$, we have*

$$\mathcal{E}_{s,t}[X_1] - \mathcal{E}_{s,t}[X_2] \leqslant \mathcal{E}^{g_\mu}_{s,t}[X_1 - X_2], \quad \text{a.s. .} \tag{18}$$

Furthermore, assume that there is $g_0 \in \mathcal{L}^2_{\mathcal{F}}(0,T)$ such that

$$\mathcal{E}^{-g_\mu+g_0}_{s,t}[0] \leqslant \mathcal{E}_{s,t}[0] \leqslant \mathcal{E}^{g_\mu+g_0}_{s,t}[0].$$

Then there is a function $f : \Omega\times[0,T]\times\mathbb{R}^{d+1}\to\mathbb{R}$ which satisfies (9), such that

$$\mathcal{E}_{s,t}[\xi] = \operatorname*{esssup}_{\alpha\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^{d+1})} \operatorname*{essinf}_{\beta\in V_{d+1}} E\left[\Gamma^{\beta}_{s,t}\xi + \int_s^t \Gamma^{\beta}_{s,r}F(r,\alpha(r),\beta(r))\,dr \;\middle|\; \mathcal{F}_s\right].$$

Here $\{\Gamma^{\beta}_{t,s}; t\leqslant s\leqslant T\}$ is the solution of SDE (15) and the function $F : \Omega\times[0,T]\times\mathbb{R}^{d+1}\times\mathbb{R}^{d+1}\to\mathbb{R}$ is given by (12).

5 Reflected backward stochastic differential equations and related mixed stochastic differential games

We make the following assumption.

(B3) The obstacle $\{S_t, 0\leqslant t\leqslant T\}$ is a continuous progressively measurable real-valued process satisfying

$$E\sup_{0\leqslant t\leqslant T}(S_t^+)^2 < \infty, \quad S_T \leqslant \xi, \text{a.s. .} \tag{19}$$

Consider the following RBSDE:

$$\left\{\begin{array}{rcl} dy_t &=& -f(t,y_t,z_t)\,dt - da_t + \langle z_t, dB_t\rangle; \\ y_T &=& \xi\in L^2(\mathcal{F}_T); \quad y_t \geqslant S_t, \text{a.s. } \forall t\in[0,T]; \quad \displaystyle\int_0^T (y_t - S_t)da_t = 0. \end{array}\right. \tag{20}$$

In view of [3, Theorem 5.2, page 718], it has a unique solution (y,z,a).

Given $\alpha \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^{d+1})$ and $\beta \in V_{d+1}$, identically as in Section 3, consider the function F given by (12) and the related RBSDE:

$$\left\{\begin{array}{rcl} dY_s &=& -[C\beta_1(s)Y_s + C\langle\beta_2(s), Z_s\rangle \\ && +F(s,\beta_1(s),\beta_2(s),\alpha_1(s),\alpha_2(s))]\,ds - dA_s + \langle Z_s, dB_s\rangle; \\ Y_T &=& \xi \in L^2(\mathcal{F}_T); \quad y_s \geqslant S_s, \text{a.s. } \forall s \in [0,T]; \\ && \displaystyle\int_0^T (Y_s - S_s)dA_s = 0. \end{array}\right. \tag{21}$$

The unique solution is denoted by $(Y^{\alpha,\beta}, Z^{\alpha,\beta}, A^{\alpha,\beta})$. We have for any $t \in [0,T]$,

$$\begin{aligned} Y_t^{\alpha,\beta} &= \operatorname*{esssup}_{\tau\in\mathcal{T}_t} E\Bigg[\int_t^\tau \Gamma^\beta_{t,s} F(s,\beta_1(s),\beta_2(s),\alpha_1(s),\alpha_2(s))\,ds \\ &\qquad +\Gamma^\beta_{t,\tau} S_\tau \chi_{\{\tau<T\}} + \Gamma^\beta_{t,\tau}\xi\chi_{\{\tau=T\}} \,\Bigg|\, \mathcal{F}_t\Bigg]. \end{aligned} \tag{22}$$

Theorem 5.1. *Assume that the function f satisfies (9) and the obstacle $\{S_t, 0 \leqslant t \leqslant T\}$ satisfies assumption (B3). Let (y,z,a) be the adapted solution of RBSDE (20) and $\{\Gamma^\beta_{t,s}; t \leqslant s \leqslant T\}$ the solution of SDE (15). Then we have for any $t \in [0,T]$,*

$$\begin{aligned} y_t = \operatorname*{esssup}_{\alpha\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^{d+1}),\tau\in\mathcal{T}_t} \operatorname*{essinf}_{\beta\in V_{d+1}} E\Bigg[\int_t^\tau \Gamma^\beta_{t,s} F(s,\beta_1(s),\beta_2(s),\alpha_1(s),\alpha_2(s))\,ds \\ +\Gamma^\beta_{t,\tau} S_\tau \chi_{\{\tau<T\}} + \Gamma^\beta_{t,\tau}\xi\chi_{\{\tau=T\}} \,\Bigg|\, \mathcal{F}_t\Bigg]. \end{aligned} \tag{23}$$

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